Musical Acoustics

Juhan Puhm

In many ways this is the first chapter of the “Compendium Musica”. More than three and a half decades ago my high school Algebra independent study was titled “Musical Acoustics (Intervals, Scales, Tuning and Temperament)”. This gives some indication on how long I have been with this topic. On re-reading this project I was surprised by a number of its astute observations and thought it would make a simple introduction to the entire “Compendium Musica”. It doesn’t go greatly into depth and covers briefly many topics. The last few years of exhaustive and comprehensive work on the “Compendium Musica” are in a way a fulfillment of this early assignment.

The original elucidation of the materials is a bit messy. For the sake of clarity I will use this early project as a template and rewrite as necessary. Simplicity of exposition will take priority over fidelity to the original. I will though try to retain the spirit and innocence of the original!

~~~~~~

Musical Acoustics

Intervals, Scales, Tuning
and Temperament

Introduction

The art of music is about as old as civilization itself. When we first started to hum or sing, or tap out simple rhythms with rock or stone we produced music. Obviously our first “instrument” was our voice. And so followed drums, rattles, sticks, stones and then the simple reed flutes and simple string instruments. The first of these string instruments was probably a very close relation to the bow and arrow.

However we can’t be given credit for the invention of music. The very aspect of music is itself a part of nature. A bird singing in a tree, or the simple rhythmic repetition of waves on the shore, or the thunderclap (which is imitated in our drums), or simply the howling of the wind, all can be attributed to nature. And so nature’s music can be connected to the beginning of time on this planet. Of all the arts music must be the closest to nature. Its direct link to nature is proof of that.

For any sound to be heard or produced vibrations must take place. A musical tone is a simple controlled vibration of a medium at a given frequency following a few fundamental principles. Through the medium of air it is transferred until these vibrations reach our ear, where they are translated for the brain to hear.

Not as generally realized, the science of music is also very old and distinguished. Before 500 BC the famous Greek scholar Pythagoras, known for his geometric theory, began a study of music. This is one of the first known explorations into the realm of musical acoustics. Pythagoras, as will be discussed later, determined under what conditions two musical tones sounding together produced pleasing combinations known as consonances. Throughout the centuries, great philosophers, physicists and mathematicians like Aristotle, Ptolemy,
~~~~~~

Huygens, Euler, Ohm, Young and Helmholtz have contributed to the study of musical acoustics. Of all the physicists Herman Helmholtz is probably the best known for his research and success in the study of musical acoustics. Before the turn of the 20th century Helmholtz published a book which is now basically the standard used in all studies of musical acoustics. The book “On the Sensations of Tone,” published in 1885, is said to be the “magnum opus of one of the last great Universalists of science”.

Of dozens of different topics like; wave propagation, room acoustics, resonance, intensity, tone quality etc where music is related to mathematics, I have chosen to study the area of Intervals, Scales, Tuning and Temperament. This topic deals with tunings over the centuries and why these several times have changed for more suitable ones. Throughout the centuries as music advanced, different scales were need to meet the growing demands of the music. The piano scale as we know it today was accepted only less than 200 years ago. Before that several changes in scale and interval structure took place. In the next pages I will mathematically describe these changes.

Pythagoras and the Monochord

As stated in the introduction, the famous Greek scholar Pytharoras was the first to delve into the world of musical acoustics. It should be stated here before continuing any further, that as long as we confine our music to a single melody it does not matter how we choose our scale or frequency. It is only when a combination of notes occur together that we must take into account how they sound together. When two or more notes of different frequencies occur together and have a pleasing sound they are called consonant. This is where Pythagoras made his first observations. Pythagoras derived a simple instrument that with a movable bridge divided a string into any ratio. He could therefore determine which ratios of string lengths produced pleasing or consonant intervals. This instrument is called a Monochord. Pythagoras found the simple string length ratios of 1:1 (unison), 1:2 (octave), 2:3 (perfect fifth) and 3:4 (perfect fourth) to be pleasing and consonant. These are all known as Pythagorean intervals.

If we keep multiplying the note **C** by a 3/2 perfect fifth both upwards and downwards we end up simply with what can be called a Pythagorean scale:

Gb	**Db**	**Ab**	**Eb**	**Bb**	**F**	**C**	**G**	**D**	**A**	**E**	**B**	**F#**
1024/729	256/243	128/81	32/27	16/9	4/3	1/1	3/2	9/8	27/16	81/64	243/128	729/512

If we take five notes from the above scale, **C G D A E** we get the Pentatonic scale **C D E G A** which has been known since time immemorial by many ancient cultures and civilizations.

If we take seven notes from the above scale, **F C G D A E B** we get the Diatonic scale **C D E F G A B C** which is the next most familiar and natural scale possible. We can see the intervals of this scale to be:

C	**D**	**E**	**F**	**G**	**A**	**B**	**C**
1/1	9/8	81/64	4/3	3/2	27/16	243/128	2/1

9/8   9/8   256/243   9/8   9/8   9/8   256/243

Our scale then has two distinct intervals: the Pythagorean Whole Tone: 9/8 = 1.125 and the Pythagorean Diatonic Semitone: 256/243 = 1.053

By continuing to add notes to our scale we will start adding the chromatic or black notes in between the Pythagorean whole tones. Things can get a little messy now. We can add first the **F#** which has the ratio 729/512, between the **F** and **G**. The interval between **F#** and **G** is still the familiar Pythagorean Diatonic Semitone with the ratio 256/243. However the interval between **F** and **F#** is the ratio 2187/2048 which is known as the Pythagorean Chromatic Semitone.

We can also see above with the full scale made from perfect fifths, that when we get around to where we started from, the two enharmonic notes **Gb** and **F**# are not the same pitch. They are close but not exact. This inexact closeness is known as a “comma”. It comes about mathematically as there is no way (3^x / 2^y) can ever equal 2, or to put it another way, no matter how many times we multiply the 3/2 perfect fifth together (adjusting down the octave when we exceed the octave) we will never be able to exactly hit the 2/1octave. This small interval between **Gb** (1024/729) and **F#** (729/512) is known as a Pythagorean or Ditonic comma. It has the ratio 531441/524258 which is 23.46 cents in size. That is, when we go right around the circle fifths, when we get back to the starting note it will be almost a quarter of a semitone sharp.

Finally we can see that the Pythagorean major third, which is also known as a Ditone, is quite sharp compared to the Just and pleasing major third that has the ratio 5/4. The difference between the sharp Pythagorean major third (81/64) and the Just major third (5/4) is the ratio (81/80). This is known as the Syntonic comma and it is 21.51 cents in size. That means the Pythagorean major third or Ditone is $1/5^{th}$ of a semitone too sharp and sounds harsh and out of tune.

The same thing happens with the Pythagorean minor third which has the ratio 32/27. It is itself a Syntonic comma or the ratio (81/80) flatter than the pleasing Just minor third which has the ratio 6/5.

The Pythagorean scale works great if we limit our harmony to perfect fourths and fifths, like the practice in the Middle Ages of singing the chants of the church in parallel perfect fourths and fifths, which is called Organum. However, as the practice of harmony evolved to include major and minor thirds something had to be done about the unpleasant Pythagorean major third and very depressed minor third. Eventually a compromise was necessary.

Meantone Scale

A solution to play music that has both good sounding major thirds and perfect fifths was a long time coming. In 1523 Pietro Aron proposes what is call “1/4 Syntonic Comma Meantone Temperament”. We have already seen how in Pythagorean tuning the major third or Ditone is the very sharp 81/64 interval made of four consecutive 3/2 perfect fifths: **C G D A E**. This 81/64 ratio is an 81/80 ratio sharper than a Just 5/4 major third. Again, the 81/80 ratio is called a Syntonic comma and is 21.51 cents in size.

Pietro Aron’s solution was to squash each of the four perfect fifths by 5.38 cents each, or 1/4 Syntonic comma, hence the name of this temperament; “1/4 Syntonic Comma Meantone Temperament”. In this temperament the major thirds remain perfectly pure or Just. The scale in perfect fifths now looks like:

Eb -¼ s.c. **Bb** -¼ s.c. **F** -¼ s.c. **C** -¼ s.c. **G** -¼ s.c. **D** -¼ s.c. **A** -¼ s.c. **E** -¼ s.c. **B** -¼ s.c. **F#** -¼ s.c. **C#** -¼ s.c. **G#** (wolf)

or arranged in major and minor thirds:

C -¼ s.c. **Eb** just **G** -¼ s.c. **Bb** just **D** -¼ s.c. **F** just **A** -¼ s.c. **C** just **E** -¼ s.c **G** just **B** -¼ s.c. **D** just **F#** -¼ s.c. **A** just **C#** -¼ s.c. **E** just **G#** -¼ s.c. **B**

We can see here not only the Just major thirds but also that each minor third is also tempered flat 1/4 Syntonic comma. Both the perfect fifth and minor third can be tempered this amount flat and yet still sound good and retain their characteristics. “1/4 Syntonic Comma Meantone Temperament” is one of the most harmonious temperaments possible for major and minor chords. It sounds excellent.

In the end, 16 of the 24 major and minor triads are excellent and we can play fully in 6 major and 6 natural minor keys. The rest of the intervals and chords that are not usable are called “wolf” intervals. The most noticeable wolf interval is between G# and Eb and is called a “wolf fifth”, even though it is really a diminished sixth. The size of this “fifth” is 35.68 cents sharp of Just, or more than a third of a semitone sharper than pure and it sounds horrible and completely out of tune.

This temperament is called “Meantone” as the major second interval is exactly half the major third interval or (5/4)^(1/2), which equals the ratio 1.18034 which is 193.16 cents in size. Another way of looking at it is that this interval is exactly the mean of the two major second intervals of 10/9 and 9/8. Hence the “mean” of the two different “tones” or major seconds.

In a Meantone temperament all notes have their own name and a G# for example, is not the same pitch as an Ab. This is why in the Renaissance it was customary to have split keys on harpsichords and organs. This expanded the number of keys a Meantone temperament could be played in. These two enharmonic pitches (Ab and G#) are 2/5ths of a semitone different in pitch!

Meantone temperaments due to their wonderful harmoniousness existed on pipe organs long after the rest of music had been supplanted by Well Temperament.

<u>The Well and Equal Tempered Scale</u>

At the beginning of the 18th century Well Temperament start to take over from Meantone Temperament. The demands of modulation to farther remote keys proved the Meantone Temperament too restrictive. In 1722 and 1744 Johann Sebastian Bach completed the first and second volumes of his “Well Tempered Clavier”. The 48 Preludes and Fugues of this collection are early examples of writing in all the 12 major and 12 minor keys.

One defining feature of Well Temperament is that it has major thirds of different sizes. The major thirds of the simpler keys are closer to pure while the major thirds of the remote keys are even worse than 12Et. This introduces what is called “key colouring”, where different keys have different qualities of being more pleasing and in tune, or more bright or depressed and out of tune.

Over the next 150 years the slightly sharp and flat major thirds of Well Temperament evened out to 12 note Equal Temperament (12Et) proper. This facilitated the complex harmony and chromaticism of the late Romantic period, Impressionism, atonality and finally jazz.

The Equal Tempered scale is the easiest to compute mathematically. It consists of the octave being equally divided into 12 equal steps or semitones. Unlike Meantone or Well Temperaments, every single like interval is exactly the same size. C to C# then is the same size as F# to G or F to F# and so on. Music can now be played equally the same in every key and we can modulate right around through the keys to where we started from. Unlike Meantone Temperament, which is incredibly harmonious, or Well Temperament, where some keys are better than others, Equal Temperament is equally out of tune everywhere. This out of tuneness is diminished by the soft bell like tones of the piano which lack strong overtones. It is actually surprising how good 12Et sounds on a piano, where on a harpsichord or organ it is incredibly sharp, grating and harsh.

It is very easy to compute the ratios of Equal Temperament. Since we need to have 12 semitones all the same size to equal the octave we can write the simple equation: **a^12 = 2** A semitone then will be: **a = 2^(1/12)** which equals the ratio **1.0594631**. **f** will denote any arbitrary frequency we wish to use.

C	**C#/Db**	**D**	**D#/Eb**	**E**	**F**	**F#/Gb**	**G**	**G#/Ab**	**A**	**A#/Bb**	**B**	**C**
fa^0	**f**a^1	**f**a^2	**f**a^3	**f**a^4	**f**a^5	**f**a^6	**f**a^7	**f**a^8	**f**a^9	**f**a^10	**f**a^11	**f**a^12

A list of ratios (without **f**) wound look like: 1.000, 1.059, 1.122, 1.1890, 1.260, 1.335, 1.414, 1.498, 1.587, 1.682, 1.782, 1.888, 2.000

<u>Extra Interesting Information</u>

-Standard of Pitch

In the time of Handel and Mozart the A above middle C was tuned anywhere between 415 hz and 428 hz. The tuning of A throughout history has continually fluctuated, even as high as 461 hz. In 1953 the International Standards Organization recommended the adoption of 440 hz as the standard frequency of A above middle C. Even so, some orchestras tune to A-442 or even A-444 to achieve a little extra brightness.

Using Equal Temperament, middle C below A-440 would then be: 440 * 2^(-9/12) which equals the frequency 261.63 hz. This is different from the standard of C with the frequency of 256 hz that we find in most physics books. I am not sure where this standard for middle C originally came from!

-Overtone Series

When a string vibrates it doesn’t only vibrate as a whole but also in halves, thirds, quarters and so on. This phenomenon, where a tone is really a complex of partial tones is called the “Overtone Series”. This can be produced on a Monochord by adjusting its bridge to string length ratios of 1:1, 1:2, 1:3, 1:4, 1:5, 1:6 etc

Frequency:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
String Length:	1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	1/9	1/10	1/11	1/12	1/13	1/14	1/15	1/16

When a trumpet is blown naturally with no use of slides or valves, the above series of notes may also be attained. Of course when we hear any note we don’t hear the entire series. We are lucky if we even hear the first overtone of the octave. The Overtone Series is related mainly to the theory of tone quality of a note or pitch which depends on how strong or weak different overtones sound.

-Beats of Combined Frequencies

If two close frequencies are sounded together their difference can be heard in distinct beats. For example, when the frequencies of 400 hz and 401 hz are sounded together one can hear a distinct beat of 1hz. This is given by the simple formula: $\mathbf{F}_{beats} = \mathbf{F}_2 - \mathbf{F}_1$. If we add together and plot the two frequencies as a Sine wave on a graph we can see their constructive and deconstructive interference. The overall increase and decrease in amplitude of these two combined Sine waves is what the ear hears as beats.

Besides hearing beats on near unison notes, it is also possible to hear beats between other intervals that are slightly detuned from Just:

Octave: $\mathbf{F}_{beats} = (1)\mathbf{F}_2 - (2)\mathbf{F}_1$
Perfect Fourth: $\mathbf{F}_{beats} = (3)\mathbf{F}_2 - (4)\mathbf{F}_1$
Perfect Fifth: $\mathbf{F}_{beats} = (3)\mathbf{F}_1 - (2)\mathbf{F}_2$
Major Third: $\mathbf{F}_{beats} = (4)\mathbf{F}_2 - (5)\mathbf{F}_1$
Minor Third: $\mathbf{F}_{beats} = (6)\mathbf{F}_1 - (5)\mathbf{F}_2$
Major Sixth: $\mathbf{F}_{beats} = (3)\mathbf{F}_2 - (5)\mathbf{F}_1$
Minor Third: $\mathbf{F}_{beats} = (8)\mathbf{F}_1 - (5)\mathbf{F}_2$

and so on.

Closing Statement

In the preceding pages we have looked at a few aspects of music and mathematics. Music is unique in the arts precisely due to its ephemeral nature and how thoroughly it can be described mathematically. In this study the development of scales and their mathematical relationships has been explored. There are many other avenues that can be explored as well, like resonance, intensity and tone quality to name a few. In music there is still much to discover and explore. All has not yet been figured out. Many arguments can arise even to this day, for example, on how to mathematically define consonance and dissonance. And so, music is bound to be an art that will last for eternity!

<u>Original Bibliography</u>

Backus, John. *The Acoustical Foundations of Music.* W.W. Norton and Company Inc.

Helmholtz, Herman. *On the Sensations of Tone*. Dover Publications Inc N.Y.

Levarie, Siegmund and Levy, Ernest. *A Study in Musical Acoustics.* Kent State University Press

Lloyd, Ll. S. M*usic and Sound.* Greenwood Press Publishers

<u>Teacher's Comment and Mark!</u>

"A very interesting project. Your conclusion is appreciated. Future work may not be accepted (university level) if hand written" Mark: **18/20**